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Pawley Multiple Antisymmetry Three-Dimensional Space Groups $G_3^{l,p'}$*

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Abstract

By use of the antisymmetric characteristic method, Pawley multiple antisymmetry three-dimensional space groups $G_3^{l,p'}$ $(p=3,4,6)$ are derived.

Introduction

Crystallographic (p')-symmetry three-dimensional space groups (or Pawley colored symmetry groups) $G_3^{p'}$ $(p=3,4,6)$ were derived by Palistrant (1980, 1981), Zamorzaev, Galyarskii & Palistrant (1978) and Zamorzaev, Karpova, Lungu & Palistrant (1986). From 73 symmorphic space groups G_3 were derived 670 junior $G_3^{p'}$ $(96\ G_3^{3'}+266\ G_3^{4'}+308\ G_3^{6'})$, from 54 hemisymmorphic G_3 were derived 562 junior $G_3^{p'}$ $(75\ G_3^{3'}+252\ G_3^{4'}+235\ G_3^{6'})$ and from 103 asymmorphic G_3 were derived 980 junior $G_3^{p'}$ $(138\ G_3^{3'}+432\ G_3^{4'}+410\ G_3^{6'})$; this means that the category $G_3^{p'}$ $(p=3,4,6)$ consists of 2212 junior groups $(309\ G_3^{3'}+950\ G_3^{4'}+953\ G_3^{6'})$. By the use of the generalized antisymmetric characteristic (AC) method (Jablan, 1987, 1990, 1992*a*, *b*) all crystallographic $(p', 2^l)$-symmetry three-dimensional space groups $G_3^{l,p'}$ $(p=3,4,6)$ will be derived.

1. Some general remarks on (p') and $(p', 2^l)$ symmetry

Pawley (p') symmetry is a particular case of the general P symmetry with $P=D_{p(2p)}$, where $D_{p(2p)}$ is the regular dihedral permutation group, generated by the permutations $e_1=(1\ldots p)(p+1\ldots 2p)$ and $e_2=(1\ p+1)(2\ p+2)\ldots(p\ 2p)$, $p\geq 2$, satisfying the relations

$$e_1^p=e_2^2=(e_1e_2)^2=E.$$

For each p the group $D_{p(2p)}$ is irreducible.

By introducing l anti-identity transformations $e_3,\ldots,e_{l+2}$ (Zamorzaev, 1976; Zamorzaev & Palistrant, 1980) $(l\in N)$ commuting between themselves and with e_1, e_2, we have $(p', 2^l)$ symmetry, with the group $P=D_{p(2p)}\times C_2^l$.

In this work only junior groups of complete $(p', 2^l)$ symmetry will be considered. Every junior (p') symmetry group $G^{p'}$ is derived from a particular generating symmetry group G, and every junior $(p', 2^l)$ symmetry group $G^{l,p'}$ is derived from a particular junior (p') symmetry group (Zamorzaev, Galyarskii & Palistrant, 1978; Zamorzaev, Karpova, Lungu & Palistrant, 1986; Palistrant, 1981).

Theorem 1: (*a*) A $(p', 2^l)$ symmetry group $G^{l,p'}$ is the junior $(p', 2^l)$ symmetry group if all relations given in the presentation of its generating symmetry group G remain satisfied after replacing the generators of the group G by the corresponding $(p', 2^l)$ symmetry-group generators.

(*b*) A junior $(p', 2^l)$ symmetry group is called the M^m-type $(p', 2^l)$ symmetry group if it is an M^m-type group regarded as an l-multiple antisymmetry group.

(*c*) A junior M^m-type $(p', 2^l)$ symmetry group $G^{l,p'}$ has complete $(p', 2^l)$ symmetry if, for every i $(i=1,2,\ldots,l+2)$, the e_i transformation can be obtained in the group $G^{l,p'}$ as an independent $(p', 2^l)$ symmetry transformation.

If only condition (*c*) is not satisfied, $G^{l,p'}$ is the incomplete junior M^m-type $(p', 2^l)$ symmetry group.

Since the derivation of $(3', 2^l)$ symmetry groups coincides with the derivation of $(32, 2^l)$ symmetry groups (Jablan, 1992*a*) as the basis for the derivation of all crystallographic $(p', 2^l)$ symmetry groups (where $p=3,4,6$), $(4')$ and $(6')$ symmetry groups will be sufficient. The derivation will be realized by the use of the generalized AC method.

Definition 1: Let all the products of (p') symmetry generators of a group $G^{p'}$, within which every generator participates once at most, be formed and then subsets of transformations equivalent with regard to (p') symmetry be separated. The resulting system is called the antisymmetric characteristic of the group $G^{p'}$ and is denoted by AC$(G^{p'})$ (Jablan, 1987, 1990, 1992*a*, *b*).

Theorem 2: Two M^m-type $(p', 2^l)$ symmetry groups derived from the same (p') symmetry group for m fixed $(m=1,\ldots,l)$ are equal if and only if they possess equal antisymmetric characteristics.

The problem of differentiating between complete and incomplete $(p', 2^l)$-symmetry junior M^m-type groups can be solved by the use of the homomorphism of the subgroup $C_p=\{e_1\}$ of the group $D_{p(2p)}$ to the group C_2 at $p=0\pmod 2$

$$e_1^{2k-1}\to e_1,\quad e_1^{2k}\to E,\quad 1\leq k\leq (p+1)/2$$

(Jablan, 1992*a*, *b*).

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Table 1. *Catalogue of junior M^m-type $(p', 2^l)$-symmetry symmorphic three-dimensional space groups*

The numbers $N_m^{p'}$ $(p = 3, 4, 6)$ are:

$N_0^{p'} = 96\ G_3^{3'} + 266\ G_3^{4'} + 308\ G_3^{6'} = 670;$
$N_1^{p'} = 496\ G_3^{1,3'} + 2171\ G_3^{1,4'} + 2644\ G_3^{1,6'} = 5311;$
$N_2^{p'} = 4709\ G_3^{2,3'} + 24088\ g_3^{2,4'} + 38133\ G_3^{2,6'} = 66930;$
$N_3^{p'} = 71713\ G_3^{3,3'} + 273252\ G_3^{3,4'} + 666512\ G_3^{3,6'} = 1011477;$
$N_4^{p'} = 1283520\ G_3^{4,3'} + 2056320\ G_3^{4,4'} + 10321920\ G_3^{4,6'} = 13661760;$
$N_5^{p'} = 19998720\ G_3^{5,3'} = 19998720.$

	(3′)	(4′)	(6′)		(3′)	(4′)	(6′)		(3′)	(4′)	(6′)
2s	1	1	1	24s		8		48s	3	2	7
3s	1	2	3	25s		7		49s	1		1
4s	1	3	2	26s	1	4	3	50s	2		6
5s	1	2	3	27s	1	2	1	51s	3	1	3
6s	1	3	2	28s	1	6	5	52s	1	1	1
7s	2	6	10	29s	1	7	3	53s	2	2	5
8s	2	8	9	30s	1	13	5	54s	5	2	14
9s	1	4	7	31s	1	10	3	55s	3	2	7
10s	2	9	12	32s	1	10	5	56s	4	2	11
11s	1	3	3	33s	1	11	5	57s	2	2	5
12s	1	3	4	34s	1	7	3	58s	3	4	17
13s	1	6	11	35s	1	7	3	63s		1	
14s	1	6	7	36s	1	18	11	65s	1		1
15s	2	10	12	37s	1	16	7	66s	1		1
16s	1	4	3	40s	2		2	67s	1		
17s	1	4	7	41s	1		1	68s	1		1
18s	1	6	11	42s	1		1	69s	1	1	1
19s	2	15	22	43s	1	1	1	70s	1		
20s	1	6	5	44s	4	1	4	71s	1		2
21s	1	6	9	45s	6	1	6	72s	1	2	3
22s		2		46s	4	1	4	73s	1		1
23s		3		47s	2	2	5				

	(3′, 2)	(4′, 2)	(6′, 2)	$(3', 2^2)$	$(4', 2^2)$	$(6', 2^2)$
2s	1	1	1	1	1	1
3s	4	10	12	16	36	40
4s	3	5	4	6		
5s	4	6	14	22	20	64
6s	4	6	6	12		
7s	14	45	84	128	399	848
8s	16	54	66	120	264	360
9s	8	28	81	96	352	1260
10s	22	64	88	228	360	520
11s	4	6	8	12		
12s	5	17	23	42	108	156
13s	16	90	184	300	1440	3216
14s	14	64	84	168	480	672
15s	20	84	108	192	528	720
16s	6	12	12	24		
17s	14	48	84	168	384	672
18s	16	90	274	450	2340	9636
19s	38	252	420	804	4392	8016
20s	10	44	48	96	256	336
21s	18	168	228	432	4032	5376
22s		8				
24s		80			576	
25s		28				
26s	3	10	7	6		
27s	1					
28s	7	42	34	54	234	192
29s	6	28	12	24		
30s	7	80	34	54	420	192
31s	6	40	12	24		
32s	8	69	34	60	372	168
33s	7	70	34	54	378	192
34s	6	28	12	24		
35s	4	22	8	12		
36s	16	294	202	300	5040	3720
37s	14	192	84	168	1536	672
40s	4					
41s	2					

Table 1 (*cont.*)

	(3′, 2)	(4′, 2)	(6′, 2)	$(3', 2^2)$	$(4', 2^2)$	$(6', 2^2)$
42s	2					
43s	1					
44s	4					
45s	6					
46s	4					
47s	8	6	13	24		
48s	12	6	20	36		
49s	2					
50s	12		24	48		
51s	3					
52s	1					
53s	8	6	14	24		
54s	20	6	38	60		
55s	12	6	19	36		
56s	16	6	29	48		
57s	8	6	13	24		
58s	30	36	160	288	240	1104
65s	2					
66s	2					
68s	1					
69s	2					
71s	4					
72s	6	8	12	24		
73s	2					

	$(3', 2^3)$	$(4', 2^3)$	$(6', 2^3)$	$(3', 2^4)$	$(4', 2^4)$	$(6', 2^4)$	$(3', 2^5)$
2s	1						
3s	56						
5s	112						
7s	1400	3276	7616	13440			
8s	672						
9s	1516	3360	13776	20160			
10s	1680						
12s	336						
13s	5712	18144	43008	80640			
14s	1344						
15s	2688						
17s	1344						
18s	17220	77112	364224	685440	2056320	10321920	19998720
19s	16464	49392	106848	241920			
20s	672						
21s	10080	64512	86016	161280			
28s	336						
30s	336						
32s	336						
33s	336						
36s	5712	57456	45024	80640			
37s	1344						
58s	2016						

2. $(p', 2^l)$-symmetry three-dimensional space groups $G_3^{l,p'}$ $(p = 3, 4, 6)$

The original Fedorov symbols of symmorphic, hemisymmorphic and asymmorphic space groups (Koptsik, 1966; Zamorzaev 1976), international symbols (*International Tables for Crystallography*, 1987) and Zamorzaev notation are used in space-symmetry-group notation.

The application of the theoretical assumptions given above will be illustrated by examples of complete M^m-type $(p', 2^l)$-symmetry junior three-dimensional space groups $(p = 3, 4, 6)$ derived in the family with the common generating symmetry group $G = 7s$ $(P2/m)$, $\{a, b, c\}(2{:}\ m)$ with the AC: $\{m, cm\}\{2, 2a, 2b, 2ab\}$ belonging to the AC-equivalency class VII (Jablan, 1987, Table 1). At $p = 3$ we have two junior (3′) symmetry groups:

$$\{a, b, c^{(3}\}(2\colon m'),$$
$$\{a^{(3}, b, c\}(2'\colon m).$$

Because of the e_2 transformation m', the AC of the first group is of the form $\{e_2, e_2\}\{E, E, E, E\}$ and of the type $(2)(5)^1$, and the AC of the second is of the form $\{E, E\}\{e_2, e_2, e_2, e_2\}$ and of the same type $(3)(5)^1$. Hence, for both of them, $N_1=7$, $N_2=64$, $N_3=700$, $N_4=6720$ (Jablan, 1987, 1992*a*). So we have the following complete (3′, 2) symmetry groups:

$\{{}^*a, b, c^{(3}\}(2\colon m')$,	$\{a, b, {}^*c^{(3}\}(2\colon m')$,
$\{a, b, c^{(3}\}({}^*2\colon m')$,	$\{{}^*a, b, {}^*c^{(3}\}(2\colon m')$,
$\{{}^*a, b, c^{(3}\}(2\colon {}^*m')$,	$\{a, b, {}^*c^{(3}\}({}^*2\colon m')$,
$\{a, b, c^{(3}\}({}^*2\colon {}^*m')$,	$\{{}^*a^{(3}, b, c\}(2'\colon m)$,
$\{a^{(3}, b, {}^*c\}(2'\colon m)$,	$\{a^{(3}, b, c\}(2'\colon {}^*m)$,
$\{{}^*a^{(3}, b, {}^*c\}(2'\colon m)$,	$\{{}^*a^{(3}, b, c\}(2'\colon {}^*m)$,
$\{a^{(3}, b, {}^*c\}({}^*2'\colon m)$	$\{a^{(3}, b, c\}({}^*2'\colon {}^*m)$,

where the antisymmetries are denoted by an asterisk.

At $p=0\ (\mathrm{mod}\ 2)$, the form and, consequently, the type of AC($G^{p'}$) is obtained by the use of the homomorphism mentioned in § 1. By treating the six (4′) symmetry groups belonging to this family in this way, we have the following results: three of them, $\{a^{(4}, b, c\}(2'\colon m)$, $\{a^{(4}, b, c\}(2'\colon m^{(2})$ and $\{a^{(4}, b, c^{(2}\}(2'\colon m)$, possess ACs of the form $\{E, E\}\ \{e_2, e_2, e_1e_2, e_1e_2\}$ and of the type $(3)(\underline{9})$, where $(\underline{9})$ denotes the type of term $\{e_2, e_2, e_1e_2, e_1e_2\}$, which contains e_2 and e_1e_2 transformations. These transformations are nonequivalent in the sense of multiple antisymmetry, so with regard to the multiple antisymmetry the type of the term mentioned is (9). However, they are equivalent in the sense of (p') symmetry, so the type of this term is denoted by $(\underline{9})$. This is the reason why the derivation of multiple-antisymmetry groups from the (p') symmetry groups with such antisymmetric characteristics cannot be simply reduced according to the theory of multiple antisymmetry, *i.e.* by the derivation of M^m-type multiple antisymmetry groups of the $(m=3,\ldots, l+2)$ from the M^2-type groups, as was done in the case of $(p2, 2')$ symmetry groups. In all the cases when some part of the AC contains the equivalent transformations e_2 and e_1e_2, the type of this term will be underlined. From the first group $\{a^{(4}, b, c\}(2'\colon m)$, we derive $N_1[\{a^{(4}, b, c\}(2'\colon m)]=9$ junior complete M^1-type (4′, 2) symmetry groups:

$\{a^{(4}, b, c\}(2'\colon {}^*m)$ with AC: $\{e_3, e_3\}\{e_2, e_2, e_1e_2, e_1e_2\}$ of type $(3)(\underline{9})^3$;

$\{a^{(4}, b, c\}({}^*2'\colon {}^*m)$ with AC: $\{e_3, e_3\}\{e_2e_3, e_2e_3, e_1e_2e_3, e_1e_2e_3\}$ of type $(3)(\underline{9})^3$;

$\{{}^*a^{(4}, b, c\}(2'\colon {}^*m)$ with AC: $\{e_3, e_3\}\{e_2, e_2, e_1e_2e_3, e_1e_2e_3\}$ of type $(3)(\underline{9})^3$;

$\{a^{(4}, b, {}^*c\}(2'\colon m)$ with AC: $\{E, e_3\}\{e_2, e_2, e_1e_2, e_1e_2\}$ of type $(4)(\underline{9})^3$;

$\{a^{(4}, b, {}^*c\}({}^*2'\colon m)$ with AC: $\{E, e_3\}\{e_2e_3, e_2e_3, e_1e_2e_3, e_1e_2e_3\}$ of type $(4)(\underline{9})^3$;

$\{{}^*a^{(4}, b, {}^*c\}(2'\colon m)$ with AC: $\{E, e_3\}\{e_2, e_2, e_1e_2e_3, e_1e_2e_3\}$ of type $(4)(9)^3$;

$\{a^{(4}, {}^*b, c\}(2'\colon m)$ with AC: $\{e_3, e_3\}\{e_2, e_1e_2, e_2e_3, e_1e_2e_3\}$ of type $(3)(\underline{16})^3$;

$\{a^{(4}, {}^*b, c\}(2'\colon {}^*m)$ with AC: $\{e_3, e_3\}\{e_2, e_1e_2, e_2e_3, e_1e_2e_3\}$ of type $(3)(\underline{16})^3$;

$\{a^{(4}, {}^*b, c\}(2'\colon m)$ with AC: $\{E, e_3\}\{e_2, e_1e_2, e_1e_2e_3, e_2e_3\}$ of type $(4)(\underline{16})^3$.

From the groups with the AC of type $(3)(\underline{9})^3$ can be derived the six M^4-type groups: two of type $(4)(\underline{9})^4$, one of type $(4)(9)^4$, two of type $(3)(\underline{16})^4$ and one of type $(4)(\underline{16})^4$; from the group with the AC of type $(3)(9)^3$, the seven M^4-type groups: four of type $(4)(9)^4$, two of type $(3)(16)^4$ and one of type $(4)(16)^4$; from the groups with the AC of type $(4)(\underline{9})^3$, the ten M^4-type groups: four of type $(4)(\underline{9})^4$, two of type $(4)(9)^4$ and four of type $(4)(\underline{16})^4$; from the group with the AC of type $(4)(9)^3$, the 12 M^4-type groups: eight of type $(4)(9)^4$ and four of type $(4)(16)^4$; from the group with the AC of the type $(3)(\underline{16})^3$, the 12 M^4-type groups: four of type $(3)(\underline{16})^4$, two of type $(3)(16)^4$, four of type $(4)(\underline{16})^4$ and two of type $(4)(16)^4$; from the group with the AC of the type $(4)(\underline{16})^3$, the 18 M^4-type groups: 12 of type $(4)(\underline{16})^4$ and 6 of type $(4)(16)^4$. Hence, $N_2[\{a^{(4}, b, c\}(2'\colon m)]=93$. Since 4 M^5-type groups can be derived from the groups of types $(4)(\underline{9})^4$ and $(4)(9)^4$, 6 can be derived from $(3)(\underline{16})^4$, 8 from $(3)(16)^4$, 12 from $(4)(\underline{16})^4$ and 16 from $(3)(\underline{16})^4$, $N_3[\{a^{(4}, b, c\}(2'\colon m)]=840$.

The remaining three (4′) symmetry groups $\{a, b, c^{(4}\}(2\colon m')$, $\{a, b, c^{(4}\}(2^{(2}\colon m')$ and $\{a^{(2}, b, c^{(4}\}(2\colon m')$ possess the AC of the form $\{e_2, e_1e_2\}(E, E, E, E\}$ and of type $(\underline{4})(5)^2$, where $(\underline{4})$ denotes the type of the term $\{e_2, e_1e_2\}$. In the case of (p') symmetry groups with the AC in which the term $\{e_2, e_1e_2\}$ occurs once and only once, the series of numbers $N_m^{p'}$ can be simply computed using the following theorem.

Theorem 3: Assume that in the AC($G^{p'}$) the term $\{e_2, e_1e_2\}$ occurs once and only once. If N_m denotes the number of junior M^{m+2}-type multiple antisymmetry groups derived from the AC($G^{p'}$) treated as the AC of a two-multiple antisymmetry group, then $N_m(G^{p'})=(2^m+1)N_m/2^{m+1}(m=1,\ldots, l)$.

Proof: Because the term $\{e_2, e_1e_2\}$ occurs once and only once in the AC($G^{p'}$), it is independent of the other part of the AC. For $m=1$ it is transformed into the four terms that differ in the sense of three-multiple antisymmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\}$, $\{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$, resulting in the three terms that differ in the sense of $(p', 2)$ symmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\}=\{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$. Hence, $N_1(G^{2'})=3N_1/4$. Proceeding in the same way, for

Table 2. *Catalogue of junior M^m-type $(p', 2^l)$-symmetry hemisymmorphic three-dimensional space groups*

The numbers $N_m^{p'}$ $(p = 3, 4, 6)$ are:

$N_0^{p'} = 75\ G_3^{3'} + 252\ G_3^{4'} + 235\ G_3^{6'} = 562;$

$N_1^{p'} = 413\ G_3^{1,3'} + 1705\ G_3^{1,4'} + 1863\ G_3^{1,6'} = 3981;$

$N_2^{p'} = 3498\ G_3^{2,3'} + 13368\ G_3^{2,4'} + 19786\ G_3^{2,6'} = 36652;$

$N_3^{p'} = 37884\ G_3^{3,3'} + 88032\ G_3^{3,4'} + 180096\ G_3^{3,6'} = 306012;$

$N_4^{p'} = 362880\ G_3^{4,3'} = 362880.$

	(3′)	(4′)	(6′)		(3′)	(4′)	(6′)		(3′)	(4′)	(6′)
1*h*	1	2	3	19*h*	1	3	3	37*h*	1	10	3
2*h*	1	2	1	20*h*	2	11	12	38*h*	1	16	7
3*h*	2	7	9	21*h*	2	14	22	39*h*	2		
4*h*	2	5	5	22*h*	2	12	12	40*h*	1		
5*h*	1	3	5	23*h*	2	9	12	41*h*	1		
6*h*	2	7	12	24*h*	1	2	2	42*h*	2	1	2
7*h*	2	7	5	25*h*		4		43*h*	3	1	3
8*h*	1	2	2	26*h*		4		44*h*	2		2
9*h*	1	2	3	27*h*		2		45*h*	3	1	3
10*h*	1	4	3	28*h*		5		46*h*	4	1	4
11*h*	2	9	11	29*h*	1	7	3	47*h*	2	1	2
12*h*	2	9	6	30*h*	1	6	3	48*h*	3	2	9
13*h*	2	7	5	31*h*	1	6	2	51*h*	1		
14*h*	2	3	6	32*h*	1	6	3	52*h*	1		
15*h*	1	3	3	33*h*	1	4	2	53*h*	1		2
16*h*	1	1	1	34*h*	1	6	3	54*h*	1		1
17*h*	2	10	16	35*h*	1	11	5				
18*h*	2	8	10	36*h*	1	16	7				

	(3′, 2)	(4′, 2)	(6′, 2)	(3′, 2^2)	(4′, 2^2)	(6′, 2^2)
1*h*	3	6	7	6		
2*h*	1					
3*h*	12	35	50	72	132	212
4*h*	8	16	14	24		
5*h*	7	18	34	54	90	192
6*h*	20	60	102	192	384	648
7*h*	8	18	13	24		
8*h*	3	2	4	6		
9*h*	6	8	12	24		
10*h*	6	12	12	24		
11*h*	20	76	94	192	480	600
12*h*	12	36	24	48		
13*h*	12	28	20	48		
14*h*	12	12	24	48		
15*h*	6	10	12	24		
16*h*	2					
17*h*	26	122	242	456	1872	4074
18*h*	17	53	78	150	288	498
19*h*	5	9	15	24	18	58
20*h*	24	104	126	264	720	936
21*h*	36	256	444	768	5088	8784
22*h*	22	120	124	252	960	940
23*h*	24	88	126	264	624	948
24*h*	4	4	5	12		
25*h*		12				
26*h*		16				
28*h*		20				
29*h*	4	17	8	12		
30*h*	4	16	8	12		
31*h*	4	16	5	12		
32*h*	4	15	8	12		
33*h*	3	4	4	6		
34*h*	4	18	8	12		
35*h*	10	104	48	96	336	840
36*h*	10	134	60	96	840	384
37*h*	6	40	12	24		
38*h*	14	192	84	168	1536	672
42*h*	4					
43*h*	6					
44*h*	4					
45*h*	6					
46*h*	8					
47*h*	4					
48*h*	18	8	36	72		
53*h*	2					
54*h*	2					

Table 2 (*cont.*)

	(3′, 2^3)	(4′, 2^3)	(6′, 2^3)	(3′, 2^4)
3*h*	336			
5*h*	336			
6*h*	1344			
11*h*	1344			
17*h*	8568	23520	53760	120960
18*h*	1008			
19*h*	84			
20*h*	2016			
21*h*	16128	64512	126336	241920
22*h*	2016			
23*h*	2016			
35*h*	672			
36*h*	672			
38*h*	1344			

every m $(m = 2, \ldots, l)$, it is transformed into the 2^{m+1} terms that differ in the sense of $(m+2)$-multiple antisymmetry, resulting in the $2^m + 1$ terms that differ in the sense of $(p', 2^l)$ symmetry, so $N_m(G^{p'}) = (2^m + 1)N_m/2^{m+1}$.

Treated as the AC of a two-multiple antisymmetry group, the AC of the form $\{e_2, e_1e_2\}\{E, E, E, E\}$ and of type $(4)(5)^2$ gives $N_1 = 8$, $N_2 = 64$, $N_3 = 448$, so for the (4′) symmetry group $G^{4'} = \{a, b, c^{(4}\}(2\text{: } m'^{)})$ with the same AC, of type $(\underline{4})(5)^2$, $N_1(G^{4'}) = 6$, $N_2(G^{4'}) = 40$, $N_3 = (G^{4'}) = 252$. The same holds for the other two (4′) symmetry groups $\{a^{(4}, b, c\}(2^2\text{: } m'^{)})$, $\{a^{(2}, b, c^{(4}\}(2\text{: } m'^{)})$ with identical ACs. Hence, for the symmetry group $7\bar{s}$ $(P2/m)$, $N_1^{4'}(7s) = 45$, $N_2^{4'}(7s) = 399$, $N_3^{4'}(7s) = 3276$.

From the ten (6′) symmetry groups of the same family, two of them, $\{a, b, c^{(3}\}(2^{(2}\text{: } m'^{)})$ and $\{a^{(3}, b, c\}(2'^{)}\text{: } m^{(2})$ possess the AC of type $(3)(5)^2$, giving $N_1^{6'} = 5$, $N_2^{6'} = 34$, $N_3^{6'} = 234$; one, $\{a^{(2}, b, c^{(3}\}(2\text{: } m'^{)})$, possesses the AC of type $(3)(9)^2$, giving $N_1^{6'} = 11$, $N_2^{6'} = 132$, $N_3^{6'} = 1344$; one, $\{a^{(3}, b, c^{(2}\}(2'^{)}\text{: } m)$, possesses the AC of type $(4)(5)^2$, giving $N_1^{6'} = 8$, $N_2^{6'} = 64$, $N_3^{6'} = 448$; two, $\{a^{(6}, b, c\}(2'^{)}\text{: } m)$ and $\{a^{(6}, b, c\}(2'^{)}\text{: } m^{(2})$, possess the AC of type $(3)(\underline{9})^2$, giving $N_1^{6'} = 9$, $N_2^{6'} = 93$, $N_3^{6'} = 840$; three, $\{a, b, c^{(6}\}(2\text{: } m'^{)})$, $\{a, b, c^{(6}\}(2^{(2}\text{: } m'^{)})$ and $\{a^{(2}, b, c^{(6}\}(2\text{: } m'^{)})$, possess AC of type $(\underline{4})(9)^2$, giving $N_1^{6'} = 12$, $N_2^{6'} = 150$, $N_3^{6'} = 1512$; and one, $\{a^{(6}, b, c^{(2}\}(2'^{)}\text{: } m)$, possesses AC of type $(4)(\underline{9})^2$, giving $N_1^{6} = 13$, $N_2^{6'} = 168$, $N_3^{6'} = 1680$. Hence, $N_1^{6'}(7s) = 84$, $N_2^{6'}(7s) = 848$, $N_3^{6'}(7s) = 7616$.

In the same manner, the partial catalogue at all complete M^m-type $(p', 2^l)$-symmetry junior symmorphic three-dimensional space groups $G_3^{l,p'}$ $(p = 3, 4, 6)$ is realized. According to Jablan (1987), this partial catalogue leads to the possibility of a complete catalogue. The final results, according to symmorphic, hemisymmorphic and asymmorphic $(p', 2^l)$ symmetry groups are summarized in Tables 1 to 3.

3. Concluding remarks

For the junior M^m-type $(P', 2^l)$-symmetry three-dimensional space groups, the numbers $N_m^{p'}$

Table 3. *Catalogue of junior M^m-type $(p', 2^l)$-symmetry asymmorphic three-dimensional space groups*

The numbers $N_m^{p'}$ $(p = 3, 4, 6)$ are:

$N_0^{p'} = 138\ G_3^{3'} + 432\ G_3^{4'} + 410\ G_3^{6'} = 980;$
$N_1^{p'} = 725\ G_3^{1,3'} + 2485\ G_3^{1,4'} + 2781\ G_3^{1,6'} = 5991;$
$N_2^{p'} = 5184\ G_3^{2,3'} + 16208\ G_3^{2,4'} + 20906\ G_3^{2,6'} = 42298;$
$N_3^{p'} = 40600\ G_3^{3,3'} + 80640\ G_3^{3,4'} + 120960\ G_3^{3,6'} = 242200;$
$N_4^{p'} = 241920\ G_3^{4,3'} = 241920.$

	(3′)	(4′)	(6′)		(3′)	(4′)	(6′)		(3′)	(4′)	(6′)
1*a*	1	1	1	34*a*		2		67*a*	1	10	3
2*a*	2	4	5	35*a*		2		70*a*	1	1	1
3*a*	2	4	4	36*a*		7		71*a*	1	1	1
4*a*	2	6	8	37*a*		6		72*a*	2	1	2
5*a*	2	5	6	38*a*		4		73*a*	2	1	2
6*a*	1	2	3	39*a*		3		74*a*	1		
7*a*	2	4	5	40*a*	1	2	3	75*a*	1		
8*a*	1	1	1	41*a*	1	7	2	76*a*	1		
9*a*	2	6	10	42*a*	1	4	3	77*a*	1		
10*a*	2	6	5	43*a*	1	8	3	78*a*	1		
11*a*	2	4	5	44*a*	1	4	2	79*a*	2		2
12*a*	2	3	2	45*a*	1	4	2	80*a*	2		2
13*a*	2	5	6	46*a*	1	5	3	81*a*	3	1	3
14*a*	3	18	30	47*a*	1	8	5	82*a*	2	1	2
15*a*	3	13	17	48*a*	1	2	1	83*a*	2	1	2
16*a*	3	10	16	49*a*	1	2	1	84*a*	2	2	4
17*a*	3	9	9	50*a*	1	4	3	85*a*	2	2	4
18*a*	3	14	21	51*a*	1	2	1	86*a*	5	1	5
19*a*	3	11	21	52*a*	1	6	3	87*a*	3	2	9
20*a*	2	7	14	53*a*	1	3	1	88*a*	3	2	9
21*a*	1	2	7	54*a*	1	10	7	93*a*	1		
22*a*	2	6	10	55*a*	1	14	7	94*a*	1		
23*a*	3	11	17	56*a*	1	7	3	95*a*	1		
24*a*	2	9	10	57*a*	1	8	3	96*a*	1		1
25*a*	2	6	5	58*a*	1	6	3	97*a*	1		
26*a*	3	7	9	59*a*	1	6	3	98*a*	1		1
27*a*	2	5	5	60*a*	1	15	5	99*a*	1		2
28*a*	3	7	9	61*a*	1	14	5	100*a*	1		1
29*a*	1	2	3	62*a*	1	10	5	101*a*	1		1
30*a*		1		63*a*	1	10	3	102*a*	1		1
31*a*		1		64*a*	1	10	3	103*a*	1		2
32*a*		2		65*a*	1	10	3				
33*a*		2		66*a*	1	7	3				

Table 3 (*cont.*)

	(3′, 2)	(4′, 2)	(6′, 2)	(3′, 2^2)	(4′, 2^2)	(6′, 2^2)
1*a*	1	1	1	1		
2*a*	8	14	20	32	44	68
3*a*	6	6	8	12		
4*a*	11	39	49	81	216	246
5*a*	10	16	20	36		
6*a*	4	5	8	12		
7*a*	9	10	15	30		
8*a*	1					
9*a*	14	36	64	108	180	354
10*a*	8	14	13	24		
11*a*	8	10	13	24		
12*a*	4					
13*a*	12	20	24	48		
14*a*	48	324	507	900	5904	8964
15*a*	30	108	150	288	672	984
16*a*	30	84	141	288	528	924
17*a*	18	36	36	72		
18*a*	42	168	252	504	1344	2016
19*a*	42	132	252	504	1056	2016
20*a*	22	66	128	252	456	912
21*a*	10	24	84	96	192	672
22*a*	17	42	78	150	234	480
23*a*	30	92	146	288	576	936
24*a*	17	60	82	150	330	528
25*a*	10	18	17	36		
26*a*	18	28	36	72		
27*a*	10	14	17	36		
28*a*	18	28	36	72		
29*a*	6	8	12	24		
33*a*		8				
36*a*		24				
37*a*		16				
38*a*		8				
40*a*	6	8	12	24		
41*a*	4	20	6	12		
42*a*	3	4	7	6		
43*a*	4	20	8	12		
44*a*	4	12	6	12		
45*a*	4	12	6	12		
46*a*	6	20	12	24		
47*a*	7	52	32	54	288	174
48*a*	2					
49*a*	2					
50*a*	4	8	8	12		
51*a*	2					
52*a*	4	14	11	12		
53*a*	2					
54*a*	10	84	60	96	828	384
55*a*	10	116	60	96	720	384
56*a*	6	28	12	24		
57*a*	6	32	12	24		
58*a*	6	24	12	24		
59*a*	6	24	12	24		
60*a*	10	144	48	96	1536	336
61*a*	10	128	48	96	864	336
62*a*	8	60	36	60	240	192
63*a*	6	40	12	24		
64*a*	6	40	12	24		
65*a*	6	40	12	24		
66*a*	6	28	12	24		
67*a*	6	40	12	24		
70*a*	1					
71*a*	1					
72*a*	2					
73*a*	2					
79*a*	4					
80*a*	4					
81*a*	6					
82*a*	4					
83*a*	4					
84*a*	8	6	10	24		
85*a*	8	6	10	24		
86*a*	10					
87*a*	18	8	36	72		
88*a*	18	8	36	72		
96*a*	2					
98*a*	1					
99*a*	2					
100*a*	2					
101*a*	2					
102*a*	2					
103*a*	4		12	12		

	(3′, 2^3)	(4′, 2^3)	(6′, 2^3)	(3′, 2^4)
2*a*	112			
4*a*	504			
9*a*	672			
14*a*	17136	80640	120960	241920
15*a*	2016			
16*a*	2016			
18*a*	4032			
19*a*	4032			
20*a*	2016			
21*a*	672			
22*a*	1008			
23*a*	2016			
24*a*	1008			
47*a*	336			
54*a*	672			
55*a*	672			
60*a*	672			
61*a*	672			
62*a*	336			

($p = 3, 4, 6$) are:

$$N_0^{p2} = 309\ G_3^{3'} + 950\ G_3^{4'} + 953\ G_3^{6'} = 2212;$$

$$N_1^{p2} = 1634\ G_3^{1,3'} + 6361\ G_3^{1,4'} + 7288\ G_3^{1,6'} = 15283;$$

$$N_2^{p2} = 13391\ G_3^{2,3'} + 53664\ G_3^{2,4'} + 78825\ G_3^{2,6'} = 145880;$$

$$N_3^{p2} = 150197\ G_3^{3,3'} + 441924\ G_3^{3,4'} + 967568\ G_3^{3,6'} = 1559689;$$

$$N_4^{p2} = 1888320\ G_3^{4,3'} + 2056320\ G_3^{4,4'} + 10321920\ G_3^{4,6'} = 14266560;$$

$$N_5^{p2} = 19998720\ G_3^{5,3'} = 19998720.$$

The possible physical applications of the generalized colored symmetry groups derived are considered by Koptsik (1988).

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Mackay Groups

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Abstract

The number of junior Mackay groups of M^m type is calculated for different nonisomorphic antisymmetric characteristics formed by $1 \leq l \leq 4$ generators. Combinatorial relationships connecting Mackay and Zamorzaev multiple-antisymmetry groups are established.

The idea, originated by Speiser (1927) and realized by Weber (1929), of representing symmetry groups of bands by black-and-white plane diagrams was the starting point for introducing antisymmetry (Heesch, 1929). The color change white-black used as the possibility for the dimensional transition from the symmetry groups of friezes G_{21} to the symmetry groups of bands G_{321} or from the plane groups G_2 to the layer groups G_{32}, applied on Fedorov space groups G_3 to derive the hyperlayer-symmetry groups G_{43} (Heesch, 1930), was the beginning of the theory of antisymmetry. Its simple mathematical explanation is the following: if G is a discrete symmetry group with the anti-identity transformation e_1 satisfying the relationship $e_1{}^2 = E$ and commuting with every symmetry S from G, the group G^1, consisting of transformations S^1 ($S^1 = S$ or $S^1 = e_1 S$), is an antisymmetry group. The antisymmetry group G^1 can be the generating ($G_1 = G$), the senior ($G^1 = G \times C_2 = G \times \{e_1\}$) or the junior ($G^1 \simeq G$) group. Every junior antisymmetry group G^1 is uniquely defined by the generating symmetry group G and its subgroup H of index 2, the symmetry subgroup of G^1, *i.e.* by the symbol G/H ($G/H \simeq C_2 = \{e_1\}$). The anti-identity transformation e_1 can be interpreted as the change of any physical or geometrical bivalent property [*e.g.* (+ −), ($S\ N$), (convex concave) *etc.*] independent of the symmetry group G. The development of the theory of antisymmetry can be followed through the works of Shubnikov *et al.* (1964), Shubnikov & Koptsik (1974) and Zamorzaev (1976).

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